Journal of Statistical Physics, Vol. 91, Nos. 1/2, 1998

The Kac Version of the Sherrington–Kirkpatrick Model at High Temperatures

Anton Bovier[1]

Received August 5, 1997; final December 29, 1997

We study the Kac version of the Sherrington–Kirkpatrick (SK) model of a spin glass, i.e., a spin glass with long- but finite-range interaction on $\mathbb{Z}^d$ and Gaussian mean zero couplings. We prove that for all $\beta < 1$, the free energy of this model converges to that of the SK model as the range of the interaction tends to infinity. Moreover, we prove that for all temperatures for which the infinite-volume Gibbs state is unique, the free energy scaled by the square root of the volume converges to a Gaussian with variance $c_{\gamma,\beta}$, where γ^{-1} is the range of the interaction. Moreover, at least for almost all values of β, this variance tends to zero as γ goes to zero, the value in the SK model. We interpret our finding as a weak indication that at least at high temperatures, the SK model can be seen as a reasonable asymptotic model for lattice spin glasses.

KEY WORDS: Spin glasses; Kac limits; central limit theorems.

1. INTRODUCTION

One of the most disputed issues in the theory of spin-glasses is the question as to what extent the results obtained for the mean-field Sherrington–Kirkpatrick (SK) model [SK] are relevant for finite dimensional short-range spin glasses (for a recent review and interesting discussions on this issue we refer to [NS1, NS2]). From the point of view of the mathematical physicist, this question is not made easier by the fact that the most interesting results on the SK model (see [MPV]) are in themselves not mathematically rigorous. On the other hand, there has been considerable progress in understanding at least the high temperature features of the SK

[1] Weierstraß-Institut für Angewandte Analysis und Stochastik, D-10117 Berlin, Germany; e-mail: bovier@wias-berlin.de.

model [ALR, FZ], and very recently some very nice probabilistic tools have been employed that make the analysis of this phase rather easy and appealing [CN, T1, T2]. One could thus ask the rather modest question to what extent these high temperature mean-field results are related to the corresponding lattice models. In standard mean field theory, the clearest interpretation of mean field results as asymptotic results for a family of lattice models if given in terms of Kac models [KUH]. The Kac version of the SK-model has been considered already in [FZ], but was not further studied in the more recent developments. The purpose of this note is to do this, and to give some quite weak results that show that, at least in the high-temperature phase, the SK model can be seen to some extent as a limit of a family of Kac spin glasses.

Let us recall the definition the Kac-SK model. Let $\sigma_i \in \{-1, 1\}$, $i \in \mathbb{Z}^d$ be Ising spins. Let G_{ij}, $i, j \in \mathbb{Z}^d$ be a family of i.i.d. random variables with mean zero and variance 1. To avoid complications, in this paper we only consider the case where G_{ij} are Gaussian, but more general distributions could be considered. Let $J_\gamma(i)$ be a Kac–kernel, i.e., a positive function $\mathbb{Z}^d \to \mathbb{R}^+$ such that $J_\gamma(i) = \gamma^d J(\gamma i)$ where $\int d^d x\, J(x) = 1$ and J has compact support. Note that the normalization condition implies that $\sum_i J_\gamma(i) = 1 + \varepsilon(\gamma)$ where $\varepsilon(\gamma) \downarrow 0$ as $\gamma \downarrow 0$. Then the Hamiltonian of our model is defined, for any $\Lambda \subset \mathbb{Z}^d$ by

$$H_{\Lambda, \gamma}(\sigma)[\omega] = -\frac{1}{\sqrt{2}} \sum_{i, j \in \Lambda} \sqrt{J_\gamma(i-j)}\, G_{ij} \sigma_i \sigma_j \tag{1.1}$$

We define the partition function as

$$Z_{\Lambda, \gamma, \beta}[\omega] \equiv \mathbb{E}_\sigma \exp(-\beta H_{\Lambda, \gamma}(\sigma)[\omega]) \tag{1.2}$$

and the free energy as

$$f_{\Lambda, \gamma, \beta}[\omega] \equiv -\frac{1}{\beta\, |\Lambda|} \ln Z_{\Lambda, \gamma, \beta}[\omega] \tag{1.3}$$

Throughout the paper we will assume *periodic* boundary conditions for convenience, although this is not essential. We introduce the *quenched* and *annealed* free energies as $f^q_{\Lambda, \gamma, \beta} \equiv \mathbb{E} f_{\Lambda, \gamma, \beta}[\omega]$, respectively $f^a_{\Lambda, \gamma, \beta} \equiv -(1/\beta\, |\Lambda|) \ln \mathbb{E} Z_{\Lambda, \gamma, \beta}[\omega]$. As is well known (see, e.g., [Vu, GR, vE]), it follows from the sub-additive ergodic theorem that for all $\gamma > 0$, for almost all ω,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} f_{\Lambda, \gamma, \beta}[\omega] = \lim_{\Lambda \uparrow \mathbb{Z}^d} f^q_{\Lambda, \gamma, \beta} \equiv f_{\gamma, \beta} \tag{1.4}$$

where here as everywhere in the paper the limit $\Lambda \uparrow \mathbb{Z}^d$ is understood in the sense of van Hove. Then our first result can be formulated as follows:

Theorem 1. For all $\beta < 1$

$$\lim_{\gamma \downarrow 0} f_{\gamma, \beta} = -\frac{\beta}{4}, \qquad \text{a.s.} \tag{1.5}$$

Remark. From the work of [ALR] we know that the free energy of the SK model equals to $-(\beta/4)$ for $\beta \leqslant 1$. Theorem 1 thus says that in the high temperature regime, the free energy of the Kac–SK model converges to that of the SK model in the Lebowitz–Penrose limit. Theorem 1 extends a result of Fröhlich and Zegarlinski [FZ] proven for β small enough to the full high-temperature region.

Our second result concerns the fluctuations of the free energy.

Theorem 2. Let $\beta < 1$. Assume moreover that β is such that for all γ sufficiently small, the Kac-SK model has a weakly unique infinite volume Gibbs state for all $\beta' \leqslant \beta$. Then

(i) If γ is small enough, there exists a constant $c_{\gamma, \beta}$ such that

$$\sqrt{|\Lambda|}\,(f_{\Lambda, \gamma, \beta} - f^{q}_{\Lambda, \gamma, \beta}) \xrightarrow{\mathscr{D}} c_{\gamma, \beta}\, g, \qquad \text{as} \quad \Lambda \uparrow \mathbb{Z}^d \tag{1.6}$$

where g is a standard Gaussian random variable and $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution (or "in law").

(ii) For Lebesgue almost all such β

$$\lim_{\gamma \downarrow 0} c_{\gamma, \beta} = 0 \tag{1.7}$$

Remark. We expect of course that (1.6) and (1.7) holds for all $\beta < 1$.

Remark. This result must be contrasted to the corresponding result in the SK model, where $|\Lambda|\,(f_{\Lambda, \gamma, \beta}[\omega] - f^{q}_{\Lambda, \gamma, \beta})$ converges in distribution to a standard normal r.v. we see that on the lattice, for all positive γ, the fluctuations of the free energy are on a much larger scale then in the SK model, but at least on this scale they tend to zero with γ. In some sense, the fluctuation result in the SK model with the $|\Lambda|$-scaling should be considered as some overly refined estimate that one happens to be able to compute in the mean field model. On the level of the "normal" fluctuations on the scale $\sqrt{|\Lambda|}$ our theorem states that the properties in the Kac model converge to those of the SK model.

Remark. The hypothesis of uniqueness of the Gibbs state is a weak point in our theorem (strictly speaking, we need less, namely only the convergence of the finite volume state with periodic (or even some other) boundary conditions to a translation covariant infinite volume Gibbs state (see [AW] for definition and extensive discussion of this notion)). Let us recall from [FZ2] that "weak uniqueness" means that the finite-volume states with any fixed ("non-random") boundary condition converge to the same infinite volume measure, almost surely. We can only assert that it holds trivially in the one dimensional model for all temperatures. In arbitrary dimensions, it is easy to adapt the proof of Fröhlich and Zegarlinski [FZ2] of weak uniqueness of the Gibbs state at high temperatures given for potentials of the form $|x|^{-\alpha d}$ to show that there is a finite β_c, independent of γ, up to which uniqueness holds. We would of course expect that $\beta_c(\gamma) \to 1$, as $\gamma \downarrow 0$, but to prove this is beyond the scope of the present note.

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1, using a beautiful idea of Talagrand. In Section 3 we prove Theorem 2, using in part the equally beautiful ideas of Comets and Neveu.

2. PROOF OF THEOREM 1

The proof of Theorem 1 follows closely Talagrand's proof in the SK model that appeared in [T1]. It relies on three simple facts.

Lemma 2.1. For any β and γ,

$$\mathbb{E}Z_{\Lambda,\gamma,\beta} = e^{|\Lambda|(1+\varepsilon(\gamma))\beta^2/4} \tag{2.1}$$

Proof. Just compute. ▮

Lemma 2.2. For any β and γ,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{-1}{\beta^2 |\Lambda|} \ln\left(\frac{\mathbb{E}Z^2_{\Lambda,\gamma,\beta}}{[\mathbb{E}Z_{\Lambda,\gamma,\beta}]^2}\right) = F^{\mathrm{kac}}_{\gamma,\beta^2} \tag{2.2}$$

where $F^{\mathrm{kac}}_{\gamma,\beta}$ is the free energy of the Kac–Ising model (for a precise definition, see below). Moreover, if $\beta \leqslant 1$, then

$$\lim_{\gamma \downarrow 0} F^{\mathrm{kac}}_{\gamma,\beta^2} = 0 \tag{2.3}$$

Proof. Again a trivial computation shows that

$$\begin{aligned} \mathbb{E} Z^2_{\Lambda,\gamma,\beta} &= \mathbb{E}_\sigma \mathbb{E}_{\sigma'} \exp\left(\frac{\beta^2}{4} \sum_{i,j \in \Lambda} J_\gamma(i-j)(\sigma_i\sigma_j + \sigma'_i\sigma'_j)^2\right) \\ &= [\mathbb{E} Z_{\Lambda,\gamma,\beta}]^2 \, \mathbb{E}_\sigma \exp\left(\frac{\beta^2}{2} \sum_{i,j \in \Lambda} J_\gamma(i-j)\,\sigma_i\sigma_j\right) \end{aligned} \tag{2.4}$$

But the last factor is nothing but the partition function in the usual ferromagnetic Kac–Ising model at inverse temperature β^2. As is well-known (see, e.g., [Tho]), by sub-additivity the limit

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{-1}{\beta^2\,|\Lambda|} \ln \mathbb{E}_\sigma \exp\left(\frac{\beta^2}{2} \sum_{i,j \in \Lambda} J_\gamma(i-j)\,\sigma_i\sigma_j\right) = F^{\mathrm{kac}}_{\gamma,\beta^2} \tag{2.5}$$

exists (if the limit is taken in the sense of van Hove) for all d, all β and all positive γ. (2.3) follows from the Lebowitz–Penrose theorem that asserts that the free energy of the Kac–Ising model converges to that of the Curie-Weiss model together with the fact that the latter, given by $F^{\mathrm{CW}}_\beta = \inf_x (x^2/2 - \beta^{-1} \ln \cosh(\beta x))$, for $\beta \leqslant 1$ is equal to zero. This concludes the proof of Lemma 2. ∎

Lemma 2.3. For all β and all γ,

$$\mathbb{P}[\,|f_{\Lambda,\gamma,\beta} - \mathbb{E} f_{\Lambda,\gamma,\beta}| > x] \leqslant 2 \exp\left(-\frac{x^2\,|\Lambda|}{(1+\varepsilon(\gamma))^2}\right) \tag{2.6}$$

Proof. This Lemma is a simple consequence of Gaussian concentration inequalities (see [LT], Section 1.1, Eq. (1.6)) which assert that for any Lipshitz function f of Gaussian r.v.'s,

$$\mathbb{P}[\,|f - \mathbb{E} f| > x] \leqslant 2 \exp\left(-\frac{x^2}{2\,\|f\|^2_{\mathrm{Lip}}}\right) \tag{2.7}$$

where $\|f\|_{\mathrm{Lip}}$ denotes the Lipshitz-norm of f. Just note that $f_{\Lambda,\gamma,\beta}$ is Lipshitz as a function of the i.i.d. Gaussian random variables G_{ij}. Indeed, a simple estimate yields that

$$|f_{\Lambda,\gamma,\beta}[\omega]-f_{\Lambda,\gamma,\beta}[\omega']| \leqslant \frac{1+\varepsilon(\gamma)}{\sqrt{2\,|\Lambda|}}\sqrt{\sum_{i,j\in\Lambda}(G_{ij}[\omega]-G_{i,j}[\omega'])^2}$$

$$=\frac{1+\varepsilon(\gamma)}{\sqrt{2\,|\Lambda|}}\,\|G[\omega]-G[\omega']\|_2 \tag{2.8}$$

that is $\|f_{\Lambda,\gamma,\beta}\|_{\mathrm{Lip}} \leqslant (1+\varepsilon(g)/\sqrt{2\,|\Lambda|})$. Insertion into (2.7) gives (2.6). ∎

Remark. An elementary proof of (2.6) in the SK-case (which generalizes without difficulty to the present situation) can be found in [BGP]. A similar estimate also holds in the non-Gaussian case, provided the G_{ij} have finite exponential moments. We refer the interested reader to [T1, T2]. This allows to extend the validity of Theorem 1 to such random variables.

We now have all the tools ready to apply Talgrand's idea from [T1] to the Kac–SK model. The Paley-Szygmund inequality (see [T1]; the proof of this inequality is elementary) asserts that

$$\mathbb{P}\left[Z_{\Lambda,\gamma,\beta}>\frac{1}{2}\,\mathbb{E}Z_{\Lambda,\gamma,\beta}\right] \geqslant \frac{1}{4}\,\frac{[\mathbb{E}Z_{\Lambda,\gamma,\beta}]^2}{\mathbb{E}Z^2_{\Lambda,\gamma,\beta}}=\frac{1}{4}\exp(-\beta^2\,|\Lambda|\,[F^{\mathrm{kac}}_{\gamma,\beta^2}+o(1)]) \tag{2.9}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\left[Z_{\Lambda,\gamma,\beta}>\frac{1}{2}\,\mathbb{E}Z_{\Lambda,\gamma,\beta}\right]\\
&\quad=\mathbb{P}[\ln Z_{\Lambda,\gamma,\beta}-\mathbb{E}\ln Z_{\Lambda,\gamma,\beta}>\ln \mathbb{E}Z_{\Lambda,\gamma,\beta}-\mathbb{E}\ln Z_{\Lambda,\gamma,\beta}-\ln 2]\\
&\quad=\mathbb{P}\left[-f_{\Lambda,\gamma,\beta}+\mathbb{E}F_{\Lambda,\gamma,\beta}>-f^a_{\Lambda,\gamma,\beta}+f^q_{\Lambda,\gamma,\beta}-\frac{\ln 2}{\beta\,|\Lambda|}\right]\\
&\quad\leqslant 2\exp\left(-|\Lambda|\left[f^q_{\Lambda,\gamma,\beta}-f^a_{\Lambda,\gamma,\beta}-\frac{\ln 2}{\beta\,|\Lambda|}\right]^2\Big/(1+\varepsilon(\gamma))^2\right)
\end{aligned} \tag{2.10}$$

where we used that by Jensen's inequality, $\ln \mathbb{E}Z_{\Lambda,\gamma,\beta} \geqslant \mathbb{E}\ln Z_{\Lambda,\gamma,\beta}$. Comparing (2.9) with (2.10) we find

$$\left[f^q_{\Lambda,\gamma,\beta}-f^a_{\Lambda,\gamma,\beta}-\frac{\ln 2}{\beta\,|\Lambda|}\right]^2 \leqslant (1+\varepsilon(\gamma))^2\,\beta^2[F^{\mathrm{kac}}_{\gamma,\beta^2}+o(1)] \tag{2.11}$$

and so, in the limit as $\Lambda \uparrow \mathbb{Z}^d$,

$$\left| f^q_{\gamma,\beta} + \frac{\beta}{4} \right| \leqslant (1 + \varepsilon(\gamma))\, \beta \sqrt{F^{\mathrm{kac}}_{\gamma,\beta^2}} \tag{2.12}$$

Since $F^{\mathrm{kac}}_{\gamma,\beta^2}$ tends to zero with γ if $\beta \leqslant 1$, the claim of Theorem 1 follows. ∎

Remark. Note that this proof is totally different in spirit than the usual proofs of convergence of Kac free energies to the respective mean field free energies. We do not see how such a proof could work here. For that reason, we have no analogous result at low temperatures (leaving alone the problem that the existence of the free energy in the SK model is not known rigorously at low temperatures).

3. PROOF OF THEOREM 2

Theorem 2 is a more subtle result than Theorem 1, as can be seen by the hypothesis we need. One would expect that this theorem can be proven along the lines of the Comets–Neveu [CN] proof in the SK model, however, as we will see there are some notable differences.

The crucial idea in the work of Comets and Neveu is the use of martingale techniques. Moreover, due to the fact that the random couplings are chosen Gaussian, it is possible to use continuous time martingales and employ the convenient and well developed tools from stochastic calculus (see e.g. [RY]). The same is true in our problem, and the most elegant way to prove our theorem is by use of (rather basic) results from stochastic calculus. On the other hand, these techniques may not be too familiar to many physicists working on disordered systems. Thus it may be useful to explain the basic ideas of the proofs in a simple way rather to just cite theorems from the literature. In this spirit, we chose to stick to a discrete setting as far as reasonable, and to give proofs of the results needed in an elementary way. In this sense, Lemmata 3.1, 3.2, and 3.4 below are immediate applications of standard formulas in stochastic calculus, and our proofs imitate the standard proofs of these results in a particular setting.

A second reason for prefering the discrete setting is that stochastic calculus is essentially limited to the Gaussian case, whereas we expect similar results for more general coupling distributions (see for example [Co]). In such cases one may still use martingales, while the infinitesimal calculus is not available. We comment on this point at the end of this section.

The advantage of the Gaussian case is that the couplings can be represented as a sum

$$G_{ij} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} g_{ij}(k) \tag{3.1}$$

where the $g_{ij}(k)$ are all independent standard normal random variables, and n can be chosen arbitrarily. Comets and Neveu [CN] use the "infinitesimal version" of this decomposition by representing G_{ij} as a Brownian motion $B_{ij}(t)$ (at $t=1$) which corresponds to passing to the limit $n \uparrow \infty$. Using this, we may think of our $g_{ij}(k)/\sqrt{n}$ as finite increments of these Brownian motions, i.e., $g_{ij}(k)/\sqrt{n} = B_{ij}(k/n) - B_{ij}((k-1)/n)$.

According to this representation we introduce the (decreasing family of) sigma-algebras $\mathscr{F}_k \equiv \mathscr{F}_k^n = \sigma(g(k), g(k+1), \ldots, g(n))$ that are generated by all the $g_{ij}(l)$ with $l \geqslant k$. We will denote by $\mathscr{F}_t = \sigma(B(s), s \in [t, 1])$ the corresponding filtrations with respect to the Brownian motion.

Our principle task in the proof of the CLT is to compute the Laplace transform

$$L_{\Lambda,\gamma,\beta}(u) \equiv \mathbb{E} \exp(u \sqrt{|\Lambda|}\, [f_{\Lambda,\gamma,\beta} - \mathbb{E} f_{\Lambda,\gamma,\beta}]) \tag{3.2}$$

To compute $L_{\Lambda,\gamma,\beta}(u)$ we use the following representation of $f_{\Lambda,\gamma,\beta} - \mathbb{E} f_{\Lambda,\gamma,\beta}$ as a martingale difference sequence, namely

$$f_{\Lambda,\gamma,\beta} - \mathbb{E} f_{\Lambda,\gamma,\beta} = \sum_{k=1}^{n} f_{\Lambda,\gamma,\beta}(k) \tag{3.3}$$

with

$$f_{\Lambda,\gamma,\beta}(k) \equiv \mathbb{E}[f_{\Lambda,\gamma,\beta} \mid \mathscr{F}_k] - \mathbb{E}[f_{\Lambda,\gamma,\beta} \mid \mathscr{F}_{k+1}] \tag{3.4}$$

The standard trick now is to compute first

$$\tilde{L}_{\Lambda,\gamma,\beta,n}(u) = \mathbb{E} \exp\left(u \sqrt{|\Lambda|}\, [f_{\Lambda,\gamma,\beta} - \mathbb{E} f_{\Lambda,\gamma,\beta}] - |\Lambda| \frac{u^2}{2} F_n\right) \tag{3.5}$$

where $F_n = F_{n,\Lambda,\gamma,\beta}$ is the conditional variance of the martingale,

$$F_n \equiv \sum_{k=1}^{n} \mathbb{E}[f^2_{\Lambda,\gamma,\beta}(k) \mid \mathscr{F}_{k+1}] \tag{3.6}$$

Lemma 3.1. For any Λ, β, γ, u, we have that

$$\lim_{n\uparrow\infty} \tilde{L}_{\Lambda,\gamma,\beta,n}(u) = 1 \tag{3.7}$$

Proof. This is a standard result, and we just give the main idea of the proof. One writes the right hand side of (3.5) as

$$\begin{aligned}
&\mathbb{E}\exp\left(u\sqrt{|\Lambda|}\,[f_{\Lambda,\gamma,\beta}-\mathbb{E}f_{\Lambda,\gamma,\beta}]-|\Lambda|\frac{u^2}{2}F_n\right)\\
&\quad=\mathbb{E}[\mathbb{E}[\ldots\mathbb{E}[e^{u\sqrt{|\Lambda|}\,f_{\Lambda,\gamma,\beta}(1)-(u^2/2)\,|\Lambda|\,\mathbb{E}[f^2_{\Lambda,\gamma,\beta}(1)\,|\,\mathscr{F}_2]}\,|\,\mathscr{F}_2]\\
&\qquad\times e^{u\sqrt{|\Lambda|}\,f_{\Lambda,\gamma,\beta}(2)-(u^2/2)\,|\Lambda|\,\mathbb{E}[f^2_{\Lambda,\gamma,\beta}(2)\,|\,\mathscr{F}_3]}\,|\,\mathscr{F}_3]\ldots\\
&\qquad\times e^{u\sqrt{|\Lambda|}\,f_{\Lambda,\gamma,\beta}(n)-(u^2/2)\,|\Lambda|\,\mathbb{E}[f^2_{\Lambda,\gamma,\beta}(n)\,|\,\mathscr{F}_{n+1}]}\,|\,\mathscr{F}_{n+1}]
\end{aligned}\tag{3.8}$$

and to work up the conditional expectations one by one. The point is that by rather simple estimates, one sees that

$$|\mathbb{E}[e^{u\sqrt{|\Lambda|}\,f_{\Lambda,\gamma,\beta}(k)-(u^2/2)\,|\Lambda|\,\mathbb{E}[f^2_{\Lambda,\gamma,\beta}(k)\,|\,\mathscr{F}_{k+1}]}\,|\,\mathscr{F}_{k+1}]-1|\leqslant Cn^{-3/2} \tag{3.9}$$

where the constant C depends on t and Λ. From this (3.7) follows immediately. ∎

The important point is now the asymptotic representation of F_n given in the following lemma:

Lemma 3.2. For any Λ, β, γ, in distribution,

$$\lim_{n\uparrow\infty} F_n=\int_0^1 dt\,\frac{1}{|\Lambda|^2}\sum_{i,j\in\Lambda} J_\gamma(i-j)(\mathbb{E}[\langle\sigma_i\sigma_j\rangle_{\Lambda,\gamma,\beta}\,|\,\mathscr{F}_t])^2 \tag{3.10}$$

where $\langle\cdot\rangle_{\Lambda,\gamma,\beta}$ denotes the expectation with respect to the finite volume Gibbs measure where G_{ij} is replaced by $B_{ij}(1)$.

Proof. The main observation is that (we skip all indices referring to Λ, γ, β)

$$\begin{aligned}
f(k)&\equiv\mathbb{E}[f_{\Lambda,\gamma,\beta}\,|\,\mathscr{F}_k]-\mathbb{E}[f_{\Lambda,\gamma,\beta}\,|\,\mathscr{F}_{k+1}]\\
&=\mathbb{E}_{g(1)}\cdots\mathbb{E}_{g(k-1)}\mathbb{E}_{g'(k)}(f(g(1),\ldots,g(k-1),g(k),\ldots,g(n))\\
&\qquad-f(g(1),\ldots,g(k-1),g'(k),\ldots,g(n)))
\end{aligned}\tag{3.11}$$

where $\mathbb{E}_{g(k)}$ stands for the expectation w.r.t. all the variables $g_{ij}(k)$, and $g'(k)$ is an independent copy of the $g(k)$. Now by Taylor's formula

$$f(g(1),..., g(k-1), g(k),..., g(n)) - f(g(1),..., g(k-1), g'(k),..., g(n))$$
$$= \sum_{i,j} \frac{\partial f}{\partial g_{ij}(k)} (g_{ij}(k) - g'_{ij}(k)) + R_2 \tag{3.12}$$

where the second order remainder R_2 is a sum over terms of the form

$$\frac{\partial^2 f}{\partial g_{ij}(k)\, \partial g_{lm}(k)} (g_{ij}(k) - g'_{ij}(k))(g_{ml}(k) - g'_{ml}(k)) \tag{3.13}$$

with f evaluated at some intermediate point. But all what counts is that

$$\left| \frac{\partial^2 f}{\partial g_{ij}(k)\, \partial g_{lm}(k)} \right| \leqslant n^{-1} \frac{\sqrt{J_\gamma(i-j)\, J_\gamma(l-m)}}{|\Lambda|} \tag{3.14}$$

and that

$$\mathbb{E}_{g'(k)} |(g_{ij}(k) - g'_{ij}(k))(g_{ml}(k) - g'_{ml}(k))|$$
$$\leqslant c[1 + (|g_{ij}(k)| + |g_{lm}(k)|) + |g_{ij}(k)|\, |g_{ml}(k)|] \tag{3.15}$$

with some numerical constant c. On the other hand,

$$\mathbb{E}_{g'(k)} \frac{\partial f}{\partial g_{ij}(k)} (g_{ij}(k) - g'_{ij}(k)) = \frac{1}{2\beta\, |\Lambda|\, n^{1/2}} \langle \sigma_i \sqrt{J_g(i-j)}\, \sigma_j \rangle_{\Lambda, \gamma, \beta}\, g_{ij}(k) \tag{3.16}$$

Inserting this leading term and the previous bound on R_2 into the expression for F_n we arrive, after some similar steps at the representation

$$F_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{|\Lambda|^2} \sum_{i,j \in \Lambda} J_\lambda(i-j)(\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\Lambda, \gamma, \beta} \mid \mathscr{F}_{k+1}])^2 + n^{-1/2} R(|\Lambda|, \gamma, \beta, n) \tag{3.17}$$

where $R(|\Lambda|, \gamma, \beta, n)$ is bounded uniformly in n, but not in Λ. To be rid of this remainder, we are finally obliged to follow Comets and Neveu and represent G_{ij} as the value of a Brownian motion $B_{ij}(t)$ at $t = 1$. With $\mathscr{F}_t$, $t \in [0, 1]$ denoting the corresponding filtration, the formula (3.10) follows easily from (3.17). ∎

The crucial point is now that under the assumption of Theorem 2, the random variable $|\Lambda|\, F_\infty$ converges to a constant in probability, as $\Lambda \uparrow \infty$.

Lemma 3.3. Assume that γ, β are such that almost surely, the infinite volume Gibbs state is weakly unique. Then there exists a constant $c_{\gamma, \beta}$ such that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \int_0^t \frac{1}{|\Lambda|} \sum_{i, j \in \Lambda} J_\gamma(i-j)(\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\Lambda, \gamma, \beta} \mid \mathcal{F}_t])^2 = c_{\gamma, \beta}, \quad \text{in Prob.} \tag{3.18}$$

Moreover,

$$c_{\gamma, \beta} \leqslant \sum_{i \in \Lambda} J_\gamma(i)\, \mathbb{E}\langle\!\langle \sigma_0 \sigma'_0 \sigma_i \sigma'_i \rangle\!\rangle_{\infty, \gamma, \beta} \tag{3.19}$$

where $\langle\!\langle \cdot \rangle\!\rangle_{\infty, \gamma, \beta}$ denotes the expectation over two independent copies σ, σ' of the spin variables w.r.t. the (unique) infinite volume Gibbs measure.

Proof. Just write

$$\begin{aligned}
&\frac{1}{|\Lambda|} \sum_{i, j \in \Lambda} J_\gamma(i-j)(\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\Lambda, \gamma, \beta} \mid \mathcal{F}_t])^2 \\
&\quad = \frac{1}{|\Lambda|} \sum_{i, j \in \Lambda} J_\gamma(i-j)(\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\infty, \gamma, \beta} \mid \mathcal{F}_t])^2 \\
&\quad\quad + \frac{1}{|\Lambda|} \sum_{i, j \in \Lambda} J_\gamma(i-j)[(\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\Lambda, \gamma, \beta} \mid \mathcal{F}_t])^2 \\
&\quad\quad - (\mathbb{E}[\langle \sigma_i \sigma_j \rangle_{\infty, \gamma, \beta} \mid \mathcal{F}_t])^2]
\end{aligned} \tag{3.20}$$

Since the unique infinite volume Gibbs state will be translation covariant, the first term converges to a constant by the ergodic theorem. Also, the finite volume states converge weakly to the infinite volume state, from which we can deduce easily that the second term converges to zero in probability. Using the Schwartz inequality and the same argument as in (3.20) in the resulting term gives the explicit representation for the upper bound on $c_{\gamma, \beta}$. This yields the lemma. ∎

Now from Lemma 3.3 and Lemma 3.1 it follows easily (since $|\Lambda| F_\infty$ is uniformly bounded) that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} L_{\Lambda, \gamma, \beta}(u) = \exp\left(\frac{c_{\gamma, \beta}}{2} u^2\right) \tag{3.21}$$

which implies part (i) of Theorem 2. What is missing is to relate $c_{\gamma, \beta}$ to the difference between the quenched and annealed free energies. To do this the upper bound (3.19) will be important.

At this point it turns out useful to compare our procedure above to the approach taken by Comets and Neveu. Following their approach, we would define

$$\hat{B}_{ij}(k) \equiv \frac{1}{\sqrt{n}} \sum_{l=1}^{k} g_{ij}(l) \tag{3.22}$$

which can also be represented by the brownian motion $B_{ij}(k/n)$. Correspondingly one can introduce $Z_{\Lambda,\gamma,\beta}(k)$ as the partition function of the model where G_{ij} is replaced by $\hat{B}_{ij}(k)$. It is easy to see that that $\hat{Z}_\Lambda(k) \equiv Z_\Lambda(k)/\mathbb{E}Z_\Lambda(k)$ is a martingale in k, i.e., $\mathbb{E}[Z_\Lambda(k)/\mathbb{E}Z_\Lambda(k) \mid \hat{\mathscr{F}}_{k-1}] = Z_\Lambda(k-1)/\mathbb{E}Z_\Lambda(k-1)$, where we have switched to the increasing sequence of sigma-algebras $\hat{\mathscr{F}}_k \equiv \sigma(g(1), g(2),..., g(k))$. Now the idea is to "take the logarithm" of this martingale, that is to write

$$\hat{Z}_\Lambda(k) = \exp(\sqrt{|\Lambda|}\, M_\Lambda(k) - \tfrac{1}{2}\, |\Lambda|\, \langle M_\Lambda(k)\rangle) \tag{3.23}$$

where $M_\Lambda(k)$ is a martingale with zero mean and $\langle M_\Lambda(k)\rangle$ is increasing (and called the "bracket" of the martingale $M_\Lambda(k)$), and hopefully, converging to a constant as Λ tends to infinity. In the SK-model, Comets and Neveu write this formula without the $\sqrt{|\Lambda|}$ and the $|\Lambda|$ coefficients and prove that still $\langle M_\Lambda(k)\rangle$ converges in that case. Then the CLT for martingales allows them to conclude that $M_\Lambda(n)$ converges to a Gaussian and this gives the desired estimate on the fluctuation of the free energy. In our case, with $\gamma > 0$, we cannot expect such a result; and convergence of $\langle M_\Lambda(k)\rangle$ can only be hoped for with that normalization. On the other hand, with this normalization $M_\Lambda(k)$ has no immediate physical interpretation, in particular it is *not* $f_\Lambda(k) - \mathbb{E}f_\Lambda(k)$! On the other hand, the bracket of this martingale does have a nice physical interpretation, namely,

$$f^q_\Lambda(k) - f^a_\Lambda(k) = \frac{1}{2\beta}\, \mathbb{E}\langle M_\Lambda(k)\rangle \tag{3.24}$$

This follows from (3.23) by taking the log and the expectation on both sides and recalling that $M_\Lambda(k)$ has mean zero. This appears to be a most unfortunate situation: There is a physical quantity that we know to be Gaussian without control on its variance, and there is another Gaussian quantity whose variance we control nicely, but we do not know what it represents. Luckily, there is a link, due to the fact that there is also a different representation of the bracket. To see how this is derived, write (3.23) in the form

$$M_\Lambda(k) = \frac{1}{\sqrt{|\Lambda|}} \ln \frac{Z_\Lambda(k)}{\mathbb{E}Z_\Lambda(k)} + \frac{\sqrt{|\Lambda|}}{2} \langle M_\Lambda(k)\rangle \tag{3.25}$$

We want that $M_\Lambda(k)$ is a martingale, that is that $\mathbb{E}[M_\Lambda(k) \mid \hat{\mathcal{F}}_{k-1}] = M_\Lambda(k-1)$. But

$$\mathbb{E}[M_\Lambda(k) \mid \hat{\mathcal{F}}_{k-1}]$$
$$= \frac{1}{\sqrt{|\Lambda|}} \mathbb{E}[\ln \hat{Z}_\Lambda(k) \mid \hat{\mathcal{F}}_{k-1}] + \frac{\sqrt{|\Lambda|}}{2} \mathbb{E}[\langle M_\Lambda(k)\rangle \mid \hat{\mathcal{F}}_{k-1}] \tag{3.26}$$

Now

$$\mathbb{E}[\ln \hat{Z}_\Lambda(k) \mid \hat{\mathcal{F}}_{k-1}]$$
$$= \mathbb{E}[\ln \hat{Z}_\Lambda(k-1) \mid \hat{\mathcal{F}}_{k-1}] + \mathbb{E}\left[\ln\left(1 + \frac{\hat{Z}_\Lambda(k) - \hat{Z}_\Lambda(k-1)}{\hat{Z}_\Lambda(k-1)}\right) \middle| \hat{\mathcal{F}}_{k-1}\right]$$
$$= \ln \hat{Z}_\Lambda(k) + \frac{1}{2} \mathbb{E}\left[\left(\frac{\hat{Z}_\Lambda(k) - \hat{Z}_\Lambda(k-1)}{\hat{Z}_\Lambda(k-1)}\right)^2 \middle| \hat{\mathcal{F}}_{k-1}\right] + R_k$$
$$= \sqrt{|\Lambda|}\, M_\Lambda(k-1) - \frac{|\Lambda|}{2} \langle M_\Lambda(k-1)\rangle$$
$$+ \frac{1}{2} \mathbb{E}\left[\left(\frac{\hat{Z}_\Lambda(k) - \hat{Z}_\Lambda(k-1)}{\hat{Z}_\Lambda(k-1)}\right)^2 \middle| \hat{\mathcal{F}}_{k-1}\right] + R_k \tag{3.27}$$

where R_k corresponds to the third order remainder in the Taylor expansion of the logarithm and is of order $n^{-3/2}$ and therefore can be made irrelevantly small by taking n to infinity (this is completely analogous to the estimates in the first part of this section. Thus, for $M_\Lambda(k)$ to be a martingale, we must have that

$$\mathbb{E}[\langle M_\Lambda(k)\rangle \mid \hat{\mathcal{F}}_{k-1}] - \langle M_\Lambda(k-1)\rangle$$
$$\equiv \mathbb{E}[\langle M_\Lambda(k)\rangle - \langle M_\Lambda(k-1)\rangle \mid \hat{\mathcal{F}}_{k-1}]$$
$$= \frac{1}{|\Lambda|} \mathbb{E}\left[\left(\frac{\hat{Z}_\Lambda(k) - \hat{Z}_\Lambda(k-1)}{\hat{Z}_\Lambda(k-1)}\right)^2 \middle| \hat{\mathcal{F}}_{k-1}\right] + R_k \tag{3.28}$$

From this we deduce the obvious solution

$$\langle M_\Lambda(k)\rangle = \frac{1}{|\Lambda|} \sum_{l=0}^{k} \mathbb{E}\left[\left(\frac{\hat{Z}_\Lambda(l) - \hat{Z}_\Lambda(l-1)}{\hat{Z}_\Lambda(l-1)}\right)^2 \middle| \hat{\mathcal{F}}_{l-1}\right] + O(n^{-1/2}) \tag{3.29}$$

It remains to compute $\mathbb{E}[((\hat{Z}_\Lambda(l) - \hat{Z}_\Lambda(l-1))/(\hat{Z}_\Lambda(l-1))^2 \mid \hat{\mathcal{F}}_{l-1}]$. But this goes just like in the previous case, and in complete analogy to Lemma 3.3 we obtain

Lemma 3.4.

$$\langle M_\Lambda(t)\rangle \equiv \lim_{n\uparrow\infty}\langle M_\Lambda([tn])\rangle = \int_0^t \frac{1}{|\Lambda|}\sum_{i,j\in\Lambda} \langle\!\langle \sigma_i\sigma_i' J_\gamma(i-j)\,\sigma_j\sigma_j'\rangle\!\rangle_{\Lambda,\gamma,\beta}(s)\,ds \tag{3.30}$$

in distribution.

By the same arguments as before, and using that a Brownian motion $B_{ij}(t)$ has the same distribution as $\sqrt{t}\,B_{ij}(1)$, we get that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathbb{E}\langle M_\Lambda(t)\rangle = \int_0^t \sum_{i\in\Lambda}\mathbb{E}\langle\!\langle \sigma_0\sigma_0' J_\gamma(i)\,\sigma_i\sigma_i'\rangle\!\rangle_{\infty,\gamma,\beta\sqrt{s}}\,ds \tag{3.31}$$

But by (3.18) and (3.24), this yields

$$\int_0^t ds c_{\gamma,\beta\sqrt{s}} \leqslant \frac{\beta}{2}[f^q_{\beta\sqrt{t}} - f^a_{\beta\sqrt{t}}] \tag{3.32}$$

From this inequality and Theorem 1 we get the immediate corollary:

Corollary 3.5. Let $\beta_c < 1$ be such that the assumptions of Lemma 3.3 are satisfied for all $\beta \leqslant \beta_c$. Then, for Lebesgue almost all $0 \leqslant \beta \leqslant \beta_c$,

$$\lim_{\gamma\downarrow 0} c_{\gamma,\beta} = 0 \tag{3.33}$$

Proof. Note that by construction $c_{\gamma,\beta}$ is non-negative. Moreover, by Theorem 1 its integral converges to zero as $\gamma\downarrow 0$. But than $c_{\gamma,\beta}$ itself must converge to zero except on a null set. This proves the corollary. ∎

This also conclude the proof of part two of Theorem 2. ∎

Remark. Most of the analysis presented in this section can be carried over to the case of non Gaussian couplings. Namely, instead of introducing the filtrations according to the decomposition of the Gaussian (3.1) one may introduce filtrations $\mathscr{F}_{k,l} = \sigma((G_{ij}, (i,j) \geqslant (k,l))$ (with some ordering on $\mathbb{Z}^{2d}$, etc., and corresponding martingales (see e.g., [AW]). Except that greater care is then necessary when treating error terms, very little will change, with one notable exception: *we will not be able to prove the convergence of the quadratic variation F_Λ to a constant.* The reason is that the expression corresponding to (3.10) will read

$$F_\Lambda = \frac{1}{|\Lambda|^2}\sum_{i,j}\mathbb{E}[\langle\!\langle \sigma_i\sigma_i' J_\gamma(i-j)\,\sigma_j\sigma_j'\rangle\!\rangle_{\Lambda,\gamma,\beta}\,|\,\mathscr{F}_{i,j}]$$

and the conditioning is breaking the translation covariance properties that were used in the proof of Lemma 3.3.

ACKNOWLEDGMENTS

I gratefully acknowledge helpful discussions with Dmitry Ioffe and Véronique Gayrard. The comments of an anonymous referee have been helpful to improve the presentation of the paper. This work was partially supported by the Commission of the European Communities under contract CHRX-CT96-0411.

REFERENCES

[ALR] M. Aizenman, J. L. Lebowitz, and D. Ruelle, "Some rigorous results on the Sherrington–Kirkpatrick spin glass model," *Commun. Math. Phys.* **112**:3–20 (1987).

[AW] M. Aizenman and J. Wehr, Rounding effects on quenched randomness on first-order phase transitions, *Commun. Math. Phys.* **130**:489 (1990).

[BGP] A. Bovier, V. Gayrard, and P. Picco, "Gibbs states of the Hopfield model with extensively many patterns," *J. Stat. Phys.* **79**:395–414 (1995).

[CN] F. Comets and J. Neveu, "The Sherrington–Kirkpatrick model of spin glasses and stochastic calculus, the high temperature case," *Commun. Math. Phys.* **166**:549–564 (1995).

[Co] F. Comets, "The martingale method for mean-field disordered systems at high temperatures," in *Mathematical aspect of spin glasses and neural networks*, A. Bovier and P. Picco, eds., Progress in Probability, Vol. 41 (Birkhäuser, Basel-Boston, 1997).

[vEvH] A. C. D. van Enter and J. L. van Hemmen, "Thermodynamic limit for long range spin glasses," *J. Stat. Phys.* **32**:141–152 (1985).

[FZ] J. Fröhlich and B. Zegarlinski, "Some comments on the Sherrington–Kirkpatrick model of spin glasses," *Commun. Math. Phys.* **112**:553–566 (1987).

[FZ2] J. Fröhlich and B. Zegarlinski, "The high-temperature phase of long-range spin glasses," *Commun. Math. Phys.* **110**:547–560 (1982).

[GR] S. Goulart-Rosa, "The thermodynamic limit of quenched free energy of magnetic systems with random interactions," *J. Phys. A* **15**:L51–L54 (1982).

[KUH] M. Kac, G. Uhlenbeck, and P.C. Hemmer, "On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model," *J. Math. Phys.* **4**:216–228 (1963); "II. Discussion of the distribution functions," *J. Math. Phys.* **4**:229–247 (1963); "III. Discussion of the critical region," *J. Math. Phys.* **5**:60–74 (1964).

[LT] M. Ledoux and M. Talagrand, *Probability in Banach spaces* (Springer, Berlin-Heidelberg–New York, 1991).

[MPV] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin-glass theory and beyond* (World Scientific, Singapore, 1988).

[NS1] Ch. M. Newman and D. L. Stein, "Non-mean-field behaviour in realistic spin glasses," *Phys. Rev. Lett.* **76**:515–518 (1996).

[NS2] Ch. M. Newman and D. L. Stein, "Thermodynamic chaos and the structure of short range spin glasses," in *Mathematical aspects of spin glasses and neural networks*, A. Bovier and P. Picco, eds., Progress in Probability, Vol. 41 (Birkhäuser, Basel–Boston, 1997).

[SK] D. Sherrington and S. Kirkpatrick, "Solvable model of a spin glass," *Phys. Rev. Lett.* **35**:1792–1796 (1972).

[T1] M. Talagrand, "Concentration of measure and isoperimetric inequalities in product space," *Publ. Math. I.H.E.S.* **81**:73–205 (1995).

[T2] M. Talagrand, "The Sherrington–Kirkpatrick model: A challenge for mathematicians," *Prob. Theor. Rel. Fields* **110**:109–176 (1998).

[Tho] C. J. Thompson, *Classical equilibrium statistical mechanics* (Clarendon Press, Oxford, 1988).

[Vu] P. Vuillermont, "Thermodynamics of quenched random spin systems and applications to the problem of phase transitions in magnetic spin glass," *J. Phys. A* **10**:1319–1333 (1977).